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ASYMPTOTICALLY PENDULUM-LIKE MOTIONS OF THE HESS-APPEL'ROT GYROSCOPE*

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Asymptotically pendulum-like motions of a heavy rigid body whose centre of gravity lies on the perpendicular to the circular cross-section of the gyration ellipsoid (the Hess-Appel'rot gyroscope) is investigated. Lyapunov's theorem is used to show that the initial position and initial angular velocity of this gyroscope can also be chosen such, that its motion will tend asymptotically, as time increases without limit, to rotation about the horizontal axis. Since in this case the initial conditions do not satisfy the invariant Hess relation, it follows that the results described cannot be obtained by direct generalisation of /1/ where the asymptotically pendulum-like motions were obtained for the special case of the Hess solution by constructing the phase trajectories.

Various examples of asymptotic motions in the classical problem of the motion of a heavy rigid body with a fixed point are shown in /1—5/.

Let the centre of gravity of a heavy rigid body with a fixed point lie on the perpendicular to the circular cross-section of the gyration ellipsoid constructed at the fixed point. We attach to this body a special coordinate system, and write its equations of motion about the fixed point in dimensionless coordinates /6/

$$\begin{aligned} &x^{\cdot} = -xz, \quad y^{\cdot} = (a-a_2)xz + yz - \nu_2 \\ &z^{\cdot} = -(a-a_2)xy + x^2 - y^2 + \nu_1 \\ &\nu^{\cdot} = \omega_2\nu_1 - \omega_1\nu_2, \quad \nu_1^{\cdot} = \omega\nu_2 - \omega_2\nu, \quad \nu_2^{\cdot} = \omega_1\nu - \omega\nu_1 \\ &\omega = ax + y, \quad \omega_1 = a_2 y + x, \quad \omega_2 = a_2 z \end{aligned} \tag{1}$$

where x, y, z are the components of the angular momentum vector, $\omega, \omega_1, \omega_2$ are the components of the angular velocity vector, ν, ν_1, ν_2 are the components of the unit vector indicating the direction of the force of gravity, a, a_2 are the dimensionless parameters characterising the ratios of the gyration tensor components, and a dot accompanying the variable denotes differentiation with respect to time.

Equations (1) have the following first integrals:

$$\begin{aligned} &ax^2 + a_2(y^2+z^2) + 2xy - 2\nu = 2E \\ &\nu^2 + \nu_1^2 + \nu_2^2 = 1, \quad x\nu + y\nu_1 + z\nu_2 = k \end{aligned} \tag{2}$$

We shall use the same variables accompanied by an asterisk to write the particular solution of (1), describing the motion of a body about the horizontal axis, and the values of the constants of the integrals (2) of this solution. Then the solution will have the form

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$$x^* = y^* = 0, \quad z^* = \varphi^{\cdot}/a_2, \quad \nu^* = \cos\varphi, \quad \nu_1^* = -\sin\varphi, \quad \nu_2^* = 0 \tag{3}$$
$$\varphi^{\cdot} = \sqrt{\mu\cos\varphi - \lambda} \quad (\mu = 2a_2,\ \lambda = -2a_2E^*)$$

(The integration constant of the angular momentum integral is obviously zero).

The variational equations for solution (3) were studied in /6/ only from the point of view of their integration, and /7/ dealt with the problem of the Lyapunov stability of the solution in question assuming that the maximum deviation of the centre of gravity of the body from the position of stable equilibrium was small. Here we shall concern ourselves with the problem of investigating the conditions of asymptotic motions of the body which tend, as $t \to \infty$, to the motion of a physical pendulum described, in the case of its rotation by (3), i.e. when $E^* > 1$. Then from (3) we have

$$\sin\frac{\varphi}{2} = \operatorname{sn}(\rho t, k'), \quad \varphi^{\cdot} = 2\rho \operatorname{dn}(\rho t, k') \tag{4}$$
$$k' = \sqrt{\frac{2}{1+E^*}}, \quad \rho = \sqrt{\frac{\mu - \lambda}{2}}$$
$$(\sin\varphi = 2\operatorname{sn}(\rho t, k')\operatorname{cn}(\rho t, k'),\ \cos\varphi = 1 - 2\operatorname{sn}^2(\rho t, k'))$$

Let the write the equations for the perturbations. As in /7/, we obtain

$$x = x^* + x_1, \quad y = y^* + x_2, \quad z = z^* + y_1 \tag{5}$$
$$\nu = \nu^* + y_2, \quad \nu_1 = \nu_1^* + y_3, \quad \nu_2 = \nu_2^* + x_3$$

The variable φ increases monotonically with time, by virtue of (4), and for this reason we shall use it as the fundamental variable, Substituting relations (5) into (1) and denoting differentiation with respect to φ by a prime we obtain taking (3) into account.

$$x_1' = -\varkappa\left(x_1 + \frac{1}{z^*}x_1y_1\right) \tag{6}$$
$$x_2' = \varkappa\left\{(a - a_2)x_1 + x_2 - \frac{x_3}{z^*} + \frac{y_1}{z^*}[(a - a_2)x_1 + x_2]\right\}$$
$$x_3' = \frac{\varkappa}{z^*}[(\cos\varphi + a\sin\varphi)x_1 + (\sin\varphi + a_2\cos\varphi)x_2 +$$
$$y_2(x_1 + a_2x_2) - y_3(x_2 + ax_1)]$$
$$y_1' = \frac{\varkappa}{z^*}[y_3 - (a - a_2)x_1x_2 + x_1^2 - x_2^2]$$
$$y_2' = -\frac{\sin\varphi}{z^*}y_1 + y_3 + \frac{1}{z^*}y_1y_3 - \frac{\varkappa x_3}{z^*}(x_1 + a_2x_2)$$
$$y_3' = -\frac{\cos\varphi}{z^*}y_1 - y_2 - \frac{1}{z^*}y_1y_2 + \frac{\varkappa x_3}{z^*}(x_2 + ax_1), \quad \varkappa = \frac{1}{a_2}$$

Let us consider a system, corresponding to the first approximation, and denote it by $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, y_1^{(1)}, y_2^{(1)}, y_3^{(1)}$. As was said in /7/, this gives us two closed systems of differential equations. From (6) it follows that

$$x_1^{(1)\prime} = -\varkappa x_1^{(1)}, \quad x_2^{(1)\prime} = \varkappa(a - a_2)x_1^{(1)} + \varkappa x_2^{(1)} - \frac{\varkappa}{z^*}x_3^{(1)} \tag{7}$$
$$x_3^{(1)\prime} = \varkappa\frac{\cos\varphi + a\sin\varphi}{z^*}x_1^{(1)} + \varkappa\frac{\sin\varphi + a_2\cos\varphi}{z^*}x_2^{(1)}$$
$$y_1^{(1)\prime} = \frac{\varkappa}{z^*}y_3^{(1)}, \quad y_2^{(1)\prime} = -\frac{\sin\varphi}{z^*}y_1^{(1)} + y_3^{(1)}, \quad y_3^{(1)\prime} = -\frac{\cos\varphi}{z^*}y_1^{(1)} - y_2^{(1)} \tag{8}$$

Integrals (2) also generate the integrals of (7) and (8)

$$x_1^{(1)}\cos\varphi - x_2^{(1)}\sin\varphi + z^*x_3^{(1)} = c^{(1)}$$
$$\frac{z^*}{\varkappa}y_1^{(1)} - y_2^{(1)} = c^{(2)}, \quad y_2^{(1)}\cos\varphi - y_3^{(1)}\sin\varphi = c^{(3)}$$

By virtue of relation $\nu^2 + \nu_1^2 + \nu_2^2 = 1$ given in (2), the constant $c^{(3)}$ should be regarded, in the case of real motion, as zero.

Let us denote by $\mathbf{X}(\varphi) = \|x_{sj}\|$, $\mathbf{Y}(\varphi) = \|y_{sj}\|$ the fundamental matrices, by $\mathbf{x}$ a vector with coordinates x_1, x_2, x_3 and by $\mathbf{y}$ the vector with coordinates y_1, y_2, y_3. Then the general solutions of (7), (8) will be

$$\mathbf{x} = \mathbf{X}(\varphi)\mathbf{b}, \quad \mathbf{y} = \mathbf{Y}(\varphi)\mathbf{c} \tag{9}$$

where $\mathbf{b}$ is a vector with coordinates b_1, b_2, b_3, $\mathbf{c}$ a vector with coordinates c_1, c_2, c_3, and

$$x_{11} = e^{-\varkappa\varphi}, \quad x_{12} = 0, \quad x_{13} = 0, \quad x_{21} = \varkappa z^*e^{\varkappa\varphi}I_1, \quad x_{22} = \frac{z^*}{2\varkappa\rho}e^{\varkappa\varphi} \tag{10}$$
$$x_{23} = -2\varkappa\rho^2z^*e^{\varkappa\varphi}I_2, \quad x_{31} = -\frac{\cos\varphi}{z^*}e^{-\varkappa\varphi} + \varkappa\sin\varphi e^{\varkappa\varphi}I_1$$
$$x_{32} = \frac{\sin\varphi}{2\varkappa\rho}e^{\varkappa\varphi}, \quad x_{33} = 2\varkappa\rho\left(\frac{1}{z^*} - \rho e^{\varkappa\varphi}\sin\varphi I_2\right)$$
$$y_{11} = \sin\varphi, \quad y_{12} = -\frac{\varkappa}{z^*} + \varkappa^2\sin\varphi\, I_3, \quad y_{13} = -\frac{\varkappa}{z^*}\cos\varphi + \varkappa^2\sin\varphi\, I_4$$
$$y_{21} = \frac{z^*}{\varkappa}\sin\varphi, \quad y_{22} = \varkappa z^*\sin\varphi I_3, \quad y_{23} = -\cos\varphi + \varkappa z^*\sin\varphi I_4$$

$$y_{31} = \frac{z^*}{\varkappa}\cos\varphi, \quad y_{32} = \varkappa z^*\cos\varphi I_3, \quad y_{33} = \sin\varphi + \varkappa z^*\cos\varphi I_4$$

$$I_1 = \int_{\varphi_0}^{\varphi}\left(\frac{a-a_1}{z^*} + \frac{\cos\varphi}{z^{*3}}\right)e^{-\varkappa\varphi}d\varphi, \quad I_2 = \int_{\varphi_0}^{\varphi}\frac{e^{-\varkappa\varphi}}{z^{*3}}d\varphi$$

$$I_3 = \int_{\varphi_0}^{\varphi}\frac{d\varphi}{z^{*3}}, \quad I_4 = \int_{\varphi_0}^{\varphi}\frac{\cos\varphi}{z^{*3}}d\varphi$$

As we already said, since $y_2^{(1)}\cos\varphi - y_3^{(1)}\sin\varphi = 0$, the integration constant c_3 should be taken as zero. Retention of this constant in formula (9) does not affect the subsequent investigations and is convenient for applying the formal Lyapunov apparatus.

Since system (7), (8) is correct (its coefficients are functions of the period 2π), therefore we shall use Lyapunov's theorem on the existence of solutions of (6) in the form of series in increasing powers of the variables $q_s = \alpha_s e^{-\lambda_s\varphi}$, by considering the eigenvalues of the first approximation system (7), (8).

Let us denote by $\mathbf{u}$ the vector with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$. Then the general solution of the system (7), (8) will be

$$\mathbf{u}^{(1)} = b_1\mathbf{u}_1^{(1)} + b_2\mathbf{u}_2^{(1)} + b_3\mathbf{u}_3^{(1)} + c_1\mathbf{u}_4^{(1)} + c_2\mathbf{u}_5^{(1)} + c_3\mathbf{u}_6^{(1)} \tag{11}$$

$$\mathbf{u}_1^{(1)} = (x_{11}, x_{21}, x_{31}, 0, 0, 0), \quad \mathbf{u}_2^{(1)} = (x_{12}, x_{22}, x_{32}, 0, 0, 0) \tag{12}$$

$$\mathbf{u}_3^{(1)} = (x_{13}, x_{23}, x_{33}, 0, 0, 0), \quad \mathbf{u}_4^{(1)} = (0, 0, 0, y_{11}, y_{21}, y_{31})$$

$$\mathbf{u}_5^{(1)} = (0, 0, 0, y_{12}, y_{22}, y_{32}), \quad \mathbf{u}_6^{(1)} = (0, 0, 0, y_{13}, y_{23}, y_{33})$$

System (12) represents a normal system of solutions. Using the definition and properties of the Lyapunov eigenvalues and (10), we find the eigenvalues of the solutions (12); the eigenvalue of the solution $\mathbf{u}_1^{(1)}$ is $\varkappa > 0$, that of $\mathbf{u}_2^{(1)}$ is $\varkappa$, and the eigenvalues of the remaining solutions are all zero. According to the Lyapunov theorem the system of equations for the perturbations (6) has a solution of the following form:

$$x_s = \sum_{k=1}^{\infty} L_k^{(s)}(b_1 e^{-\varkappa\varphi})^k, \quad y_s = \sum_{k=1}^{\infty} M_k^{(s)}(b_1 e^{-\varkappa\varphi})^k \quad (s = 1, 2, 3) \tag{13}$$

where $L_k^{(s)}, M_k^{(s)}$ are continuous functions of φ independent of the constant b_1, whose eigenvalues are not less than zero.

The Lyapunov-Poincaré method can be used to obtain relations (13). Since in a normal system of solutions the only solution with a positive eigenvalue is $u_1^{(1)}$, we must put in (9), (11) $b_2 = b_3 = c_1 = c_2 = c_3 = 0$.

Let

$$\mathbf{x} = \mathbf{x}^{(1)} + \mathbf{x}^{(2)} + \ldots + \mathbf{x}^{(m)} + \ldots, \quad \mathbf{y} = \mathbf{y}^{(1)} + \mathbf{y}^{(2)} + \ldots + \mathbf{y}^{(m)} + \ldots \tag{14}$$

where $\mathbf{x}^{(m)}, \mathbf{y}^{(m)}$ are the m-th order terms. Then $\mathbf{y}^{(1)} = 0$, $\mathbf{x}^{(1)} = b_1(x_{11}, x_{21}, x_{31})$. Let us denote by $\mathbf{X}^{-1}(\varphi)$, $\mathbf{Y}^{-1}(\varphi)$ the matrices which are inverses of the fundamental matrices $\mathbf{X}(\varphi)$, $\mathbf{Y}(\varphi)$. Here we have

$$x_{11}^* = e^{\varkappa\varphi}, \quad x_{12}^* = 0, \quad x_{13}^* = 0, \quad x_{21}^* = -2\rho\varkappa^2 e^{\varkappa\varphi}I_1 + \tag{15}$$

$$2\rho^2\varkappa\cos\varphi I_2, \quad x_{22}^* = \frac{2\rho\varkappa e^{-\varkappa\varphi}}{z^*} - 2\rho^2\varkappa\sin\varphi I_2$$

$$x_{23}^* = 2\rho^2\varkappa z^* I_2, \quad x_{31}^* = \frac{\cos\varphi}{2\rho\varkappa}, \quad x_{32}^* = -\frac{\sin\varphi}{2\rho\varkappa}, \quad x_{33}^* = \frac{z^*}{2\rho\varkappa}$$

$$y_{11}^* = \varkappa z^* I_3, \quad y_{12}^* = \frac{\varkappa}{z^*}\sin\varphi - \varkappa^2 I_3 + \varkappa^2\cos\varphi I_4$$

$$y_{13}^* = \frac{\varkappa}{z^*}\cos\varphi - \varkappa^2\sin\varphi I_4, \quad y_{21}^* = -\frac{z^*}{\varkappa}, \quad y_{22}^* = 1, \quad y_{23}^* = 0$$

$$y_{31}^* = 0, \quad y_{32}^* = -\cos\varphi, \quad y_{33}^* = \sin\varphi, \quad z^* = \varkappa\sqrt{\mu\cos\varphi - \lambda}$$

The terms of the expansions (14) at $m > 1$ are found from the recurrent relations

$$\mathbf{x}^{(m)} = \mathbf{X}(\varphi)\int_{\varphi_0}^{\varphi}\mathbf{X}^{-1}(\varphi)\mathbf{f}^{(m-1)}(\varphi)d\varphi, \quad \mathbf{y}^{(m)} = \mathbf{Y}(\varphi)\int_{\varphi_0}^{\varphi}\mathbf{Y}^{-1}(\varphi)\mathbf{g}^{(m-1)}(\varphi)d\varphi$$

$$(\mathbf{f}^{(m-1)} = (f_1^{(m-1)}, f_2^{(m-1)}, f_3^{(m-1)}), \quad \mathbf{g}^{(m-1)} = (g_1^{(m-1)}, g_2^{(m-1)}, g_3^{(m-1)}))$$

and the elements of the matrices $\mathbf{X}^{-1}, \mathbf{Y}^{-1}$ are determined by (15).

By virtue of the initial system (6) we have the following relations:

$$f_1^{(m-1)} = -\frac{\varkappa}{z^*}\sum_{q=1}^{m-1} x_1^{(q)}y_1^{(m-q)}, \quad f_2^{(m-1)} = \frac{\varkappa}{z^*}\sum_{q=1}^{m-1} y_1^{(q)}[x_2^{(m-q)} + (a-a_2)x_1^{(m-q)}]$$

$$f_3^{(m-1)} = \frac{\varkappa}{z^*}\sum_{q=1}^{m-1}[y_2^{(q)}(x_1^{(m-q)} + a_2x_2^{(m-q)}) - y_3^{(q)}(x_2^{(m-q)} + ax_1^{(m-q)})]$$

$$g_1^{(m-1)} = \frac{\varkappa}{z^*}\sum_{q=1}^{m-1}[x_1^{(q)}x_1^{(m-q)} - x_2^{(q)}x_2^{(m-q)} + (a_2 - a)x_1^{(q)}x_2^{(m-q)}]$$

$$g_2^{(m-1)} = \frac{1}{z^*} \sum_{q=1}^{m-1} [y_1^{(q)} y_3^{(m-q)} - \varkappa x_3^{(q)} (x_1^{(m-q)} + a_2 x_2^{(m-q)})]$$

$$g_3^{(m-1)} = \frac{1}{z^*} \sum_{q=1}^{m-1} [-y_1^{(q)} y_2^{(m-q)} + \varkappa x_3^{(q)} (x_2^{(m-q)} + a x_1^{(m-q)})]$$

Since series (13) converge absolutely for all $\varphi \geqslant \varphi_0$ and $b_1 < b^*$, it follows that when $\varphi \to \infty$, $x_s \to 0$, $y_s \to 0$ $(s = 1, 2, 3)$. This means that we can always choose the initial position and initial angular velocity of the Hess-Appel'rot gyroscope in such a manner, that its motion will tend, as time increases without limit, asymptotically to rotation about a horizontal axis. Such motions are called asymptotically pendulum-like motions.

We note that the class of asymptotically pendulum-like motions of the Hess-Appel'rot gyroscope described by relations (13) does not include the Hess solution as a special case. Indeed, the latter solution for the system of differential equations (6) is characterised by the invariant relation $x_1 = 0$. By virtue of the first equation of this system we find that if the relation $x_1 = 0$ holds at the initial instant, it holds at any other instant. For the class of asymptotically pendulum-like motions of the Hess-Appel'rot gyroscope and constant b_1 is not zero, and therefore $x_1 \neq 0$.

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INTERACTION OF THIRD-ORDER RESONANCES IN PROBLEMS OF THE STABILITY OF HAMILTONIAN SYSTEMS*

L.G. KHAZIN

The problem of the stability of the equilibrium state of a neutral Hamiltonian systems (all eigenvalues of the linearization matrices are purely imaginary) is considered. A stability criterion is obtained for systems with several third-order resonances.

1. Formulation of the problem. We shall study the stability of the equilibrium state of an autonomous Hamiltonian system of equations

$$x_\alpha^{\cdot} = \frac{\partial H(x, y)}{\partial y_\alpha}; \quad y_\alpha^{\cdot} = -\frac{\partial H(x, y)}{\partial x_\alpha}, \quad \alpha = 1, \ldots, N \tag{1.1}$$

$$H(x, y) = H_2(x, y) + H_3(x, y) + \ldots$$

$$x = (x_1, \ldots, x_N); \quad y = (y_1, \ldots, y_N)$$

Here $H_k(x, y)$ denotes the homogeneous k-th degree polynomials, Γ is the linearization matrix of the system, (1.1), $\lambda(\Gamma)$ are the eigenvalues and $\operatorname{Re}\lambda(\Gamma) = 0$.

We shall use the following definitions.

The system

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